PROMYS India 2020

Application Problem Set

https://www.promys-india.org

Please attempt each of the following problems. Though they can all be solved with no more than a standard high school mathematics background, most of the problems require considerably more ingenuity than is usually expected in high school. You should keep in mind that we do not expect you to find complete solutions to all of them. Rather, we are looking to see how you approach challenging problems.

We ask that you tackle these problems by yourself. Please see below for further information on citing any sources you use in your explorations.

Here are a few suggestions:

- Think carefully about the meaning of each problem.
- Examine special cases, either through numerical examples or by drawing pictures.
- Be bold in making conjectures.
- Test your conjectures through further experimentation, and try to devise mathematical proofs to support the surviving ones.
- Can you solve special cases of a problem, or state and solve simpler but related problems?

If you think you know the answer to a question, but cannot prove that your answer is correct, tell us what kind of evidence you have found to support your belief. If you use books, articles, or websites in your explorations, be sure to cite your sources.

Be careful if you search online for help. We are interested in your ideas, not in solutions that you have found elsewhere. If you search online for a problem and find a solution (or most of a solution), it will be much harder for you to demonstrate your insight to us.

You may find that most of the problems require some patience. Do not rush through them. It is not unreasonable to spend a month or more thinking about the problems. It might be good strategy to devote most of your time to a small selection of problems which you find especially interesting. Be sure to tell us about progress you have made on problems not yet completely solved. **For each problem you solve, please justify your answer clearly and tell us how you arrived at your solution: this includes your experimentation and any thinking that led you to an argument.**

There are various tools available for typesetting mathematics on a computer. You are welcome to use one of these if you choose, or you are welcome to write your solutions by hand, or you might want to do a bit of both. We are interested in your mathematical ideas, not in your typesetting. What is important is that we can read all of your submitted work, and that you can include all of your ideas (including the ones that didn't fully work).

1. Calculate each of the following:

$$
\begin{aligned}
1^3 + 5^3 + 3^3 &= ??\\
16^3 + 50^3 + 33^3 &= ??\\
166^3 + 500^3 + 333^3 &= ??\\
1666^3 + 5000^3 + 3333^3 &= ??
\end{aligned}
$$

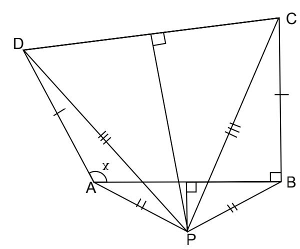
What do you see? Can you state and prove a generalization of your observations?

2. The sequence (x_n) of positive real numbers satisfies the relationship $x_{n-1}x_nx_{n+1} = 1$ for all $n \geq 2$. If $x_1 = 1$ and $x_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

The sequence (y_n) satisfies the relationship $y_{n-1}y_{n+1} + y_n = 1$ for all $n \geq 2$. If $y_1 = 1$ and $y_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

3. Consider the sequence $t_0 = 3,\ t_1 = 3^3,\ t_2 = 3^{3^3},\ t_3 = 3^{3^{3^3}},\ \ldots$ defined by $t_0 = 3$ and $t_{n+1} = 3^{t_n}$ for $n \geq 0$. What are the last two digits in $t_3 = 3^{3^{3^3}}$? Can you say what the last *three* digits are? Show that the last 10 digits of t_k are the same for all $k \geq 10$.

4. According to the Journal of Irreproducible Results, any obtuse angle is a right angle!



Here is their argument. Given the obtuse angle x, we make a quadrilateral $ABCD$ with $\angle DAB = x$, and $\angle ABC = 90°$, and $AD = BC$. Say the perpendicular bisector to DC meets the perpendicular bisector to AB at P. Then $PA = PB$ and $PC = PD$. So the triangles PAD and PBC have equal sides and are congruent. Thus $\angle PAD = \angle PBC$. But PAB is isosceles, hence $\angle PAB = \angle PBA$. Subtracting, gives $x = \angle PAD - \angle PAB = \angle PBC - \angle PBA = 90°$. This is a preposterous conclusion – just where is the mistake in the "proof" and why does the argument break down there?

5. We say that a positive integer is *quiteprime* if it is not divisible by 2, 3, or 5. How many quiteprime positive integers are there less than 100? less than 1000? A positive integer is *very quiteprime* if it is not divisible by any prime less than 15. How many very quiteprime positive integers are there less than 90000? Without giving an exact answer, can you say *approximately* how many very quiteprime positive integers are less than 10^{10}? less than 10^{100}? Explain your reasoning as carefully as you can.

6. A monkey has filled in a 3×3 grid with the numbers $1, 2, \ldots, 9$. A cat writes down the three numbers obtained by multiplying the numbers in each horizontal row. A dog writes down the three numbers obtained by multiplying the numbers in each vertical column. Can the monkey fill in the grid in such a way that the cat and dog obtain the same set of three numbers? What if the monkey writes the numbers $1, 2, \ldots, 25$ in a 5×5 grid? Or $1, 2, \ldots, 121$ in a 11×11 grid? Can you find any conditions on n that guarantee that it is possible or any conditions that guarantee that it is impossible for the monkey to write the numbers $1, 2, \ldots, n^2$ in an $n \times n$ grid so that the cat and the dog obtain the same set of numbers?

7. The set S contains some real numbers, according to the following three rules.

(i) $\frac{1}{1}$ is in S.

(ii) If $\frac{a}{b}$ is in S, where $\frac{a}{b}$ is written in lowest terms (that is, a and b have highest common factor 1), then $\frac{b}{2a}$ is in S.

(iii) If $\frac{a}{b}$ and $\frac{c}{d}$ are in S, where they are written in lowest terms, then $\frac{a+c}{b+d}$ is in S.

These rules are exhaustive: if these rules do not imply that a number is in S, then that number is not in S. Can you describe which numbers are in S?

8. Let P_0 be an equilateral triangle of area 10. Each side of P_0 is trisected, and the corners are snipped off, creating a new polygon (in fact, a hexagon) P_1. What is the area of P_1? Now repeat the process to P_1 – i.e. trisect each side and snip off the corners – to obtain a new polygon P_2. What is the area of P_2? Now repeat this process infinitely often to create an object P_∞. What is the area of P_∞?

9. The Mathematical Forest is grown in a two-dimensional plane, where trees can only grow on points with integer coordinates. To start with, there are no trees at all. The foresters plant the first tree at $(0, 0)$. Each year, they carry out tree planting according to the following rule. If there is a tree on the point (m, n) but there are no trees on the points $(m+1, n)$ and $(m, n+1)$, then they can choose to remove the tree on (m, n) and plant new trees on the points $(m, n+1)$ and $(m+1, n)$.

For an integer $k \geq 1$, the kth diagonal consists of all points (m, n) with $m + n = k - 1$. Is it possible for the foresters to arrange their planting so that eventually there are no trees on the first 2 diagonals? What about the first 3 diagonals? 4 diagonals? Can you generalize?

10. On a particularly strange railway line, there is just one infinitely long track, so overtaking is impossible. Any time a train catches up to the one in front of it, they link up to form a single train moving at the speed of the slower train. At first, there are three equally spaced trains, each moving at a different speed. After all the linking that will happen has happened, how many trains are there? What would have happened if the three equally spaced trains had started in a different order, but each train kept its same starting speed? On average (where we are averaging over all possible orderings of the three trains), how many trains will there be after a long time has elapsed? What if at the start there are 4 trains (all moving at different speeds)? Or 5? Or n? (Assume the Earth is flat and extends infinitely far in all directions.)